Published by Institute of Physics Publishing for SISSA/ISAS
Received: *January 27, 2004*
Accepted: *April 25, 2004*

Holography, dimensional reduction and the Bekenstein bound

Dongsu Bak

Physics Department, University of Seoul
Seoul 130-743, Korea
E-mail: dsbak@mach.uos.ac.kr

Ho-Ung Yee

School of Physics, Korea Institute for Advanced Study
207-43, Cheongryangri-Dong, Dongdaemun-Gu, Seoul 130-722, Korea
E-mail: ho-ung.yee@kias.re.kr

Abstract: We consider dimensional reduction of the lightlike holography of the covariant entropy bound from $D+1$ dimensional geometry of $M \times S^1$ to the D dimensional geometry M. With a warping factor, the local Bekenstein bound in $D+1$ dimensions leads to a more refined form of the bound from the D dimensional view point. With this new local Bekenstein bound, it is quite possible to saturate the lightlike holography even with nonvanishing expansion rate. With a Kaluza-Klein gauge field, the dimensional reduction implies a stronger bound where the energy momentum tensor contribution is replaced by the energy momentum tensor with the electromagnetic contribution subtracted.

Keywords: AdS-CFT and dS-CFT Correspondence, Classical Theories of Gravity, Black Holes.

Contents

1. Introduction

The holographic principle appears to be a new guiding paradigm for true understanding of quantum gravity theories. It states that the fundamental degrees of freedom in a certain region of spacetime is bounded not by its volume but by its boundary area [1, 2]. Based on the earlier attempts [3, 4], a more precise version of the holography is proposed by Bousso [5]. In this proposal, one considers the "lightsheets" consisting of lightlike geodesic of nonpositive expansion orthogonally generated from a boundary. The entropy passing through the lightsheet should be bounded by the boundary area divided by the Planck area $4G$. This statement is called the generalized covariant entropy bound (GCEB). This proposal passes many tests so far and is widely accepted by now. Later, a more refined version was given by Flanagan, Marolf and Wald [6]. Here one considers the lightlike sheets consisting of lightlike geodesics orthogonally generated from one boundary B ending on another boundary B'. Then the proposal says that the entropy passing through the lightlike sheet should be limited by the difference of the boundary areas divided by $4G$. Hence this version is certainly stronger than the original proposal by Bousso. This is the formulation of the holographic principle which we shall mainly concern in this note.

This refined version may be proved in a semiclassical context based on two conditions. One is the initial condition on the lightlike sheet saying that the minus of initial expansion rate of the geodesics divided by $4G$ should be taken to be larger than the initial entropy flux density through the lightsheet. The other is the condition on the energy flux density versus the change of the entropy flux density. This condition is kind of the local version of the Bekenstein bound of the entropy of weakly gravitating system [7],

$$S \leq \pi M d \tag{1.1}$$

where M and d are respectively the energy and the linear size of the system. From the local entropy energy condition, the Bekenstein bound may be derived straightforwardly. In this context, we shall call the local entropy-energy condition as "local Bekenstein bound".

As discussed in detail in [8], the first condition merely puts a restriction on the choice of the initial boundary surface. This is because the initial expansion rate is solely determined by the choice of the surface. Hence, at least classically, the condition that leads to the proof of the statement comes from the local condition on the entropy and energy densities. The local Bekenstein bound is clearly a sufficient condition for the holography. One unsatisfactory aspect about the condition is that there is no way to saturate the holography bound except some trivial case where the expansion remains zero all the time [8]. One typical example is the case of AdS/CFT [9] where one compares the bulk gravitational entropy with the regulated boundary area divided $4G$ corresponding to the degrees of freedom of the boundary CFT [10]. According to the above formulation of the holography, the saturation cannot occur in the example of AdS/CFT because there the expansion rate is nonvanishing for the boundary surface we are interested in [8]. This is disappointing because the conjectured AdS/CFT correspondence claims the exact equivalence at the level of Hilbert space. This problem may disappear if the local Bekenstein bound is modified appropriately, which we shall propose below.

In this note, we start from the GCEB in $D+1$ dimensions and investigate its implication upon dimensional reduction to the spacetime of the form $M \times S^1$ where S^1 is the circle and M denotes a D dimensional geometry.[1] We shall first consider the case where one has vanishing Kaluza-Klein gauge fields. Lightlike geodesics parallel to M in the higher dimensions remain lightlike viewed from lower dimensions. Assuming the local Bekenstein bound in $D+1$ dimensions, the holography in the lower dimensions should automatically follow because the dimensional reduction does not change the statement of holography except some trivial overall factor. What we are asking is then the implication of the higher dimensional local Bekenstein bound. Considering the KK reduction with a warping factor, we find that the $D+1$ dimensional local Bekenstein bound leads to a new one with an order G correction from the view point of the lower dimensional spacetime. This will be our main result. We further consider the dimensional reduction with a KK gauge field but with a trivial warp factor. This time we get a stronger version of local Bekenstein bound, which seems consistent with previous proposals in the presence of U(1) gauge field. The order G correction of the Bekenstein bound from the dimensional reduction should be considered seriously because the physics behind the Bekenstein bound is not much understood now. Another virtue of this correction is that the saturation of the bound is now possible even with nonvanishing expansion.

In section 2, we shall review the formulation of the lightlike holography of GCEB. In section 3, we consider the dimensional reduction with nontrivial warping factor or with a KK gauge field. We find interesting modifications of the local Bekenstein bound. Last section is devoted for the discussions.

[1] There are literatures [11, 12] considering dimensional reduction of holography in some other context. Ref. [13] discusses phenomenological implications of entropy bounds in scenarios of extra dimensions. See also ref. [14] for an interesting application of the holography.

JHEP04(2004)061

2. Review of covariant entropy bound

In its original form proposed by Bousso [5], the covariant entropy bound states that, given a codimension two spacelike hypersurface, the total entropy on a lightsheet generated by non expanding null geodesics orthogonal to the hypersurface should be bounded by the area of the surface in Planck unit. Null geodesics may be either future or past directed. Consider, for example, a future directed focusing lightsheet, which is orthogonal to a compact surface $\mathcal{A}$ that bounds a volume $\mathcal{V}$. The covariant entropy bound dictates the total entropy flow on the lightsheet, which is not less than the total entropy in the volume $\mathcal{V}$ due to the second law of thermodynamics, be bounded by $\mathcal{A}/4G$. Thus, the covariant entropy bound generalizes the old area bound. Another example would be taking a lightsheet generated by past-directed null geodesics that stay on the horizon of a growing black hole. The total entropy flow on this lightsheet is the total entropy of infalling matter while creating the black hole. The covariant entropy bound dictates this be bounded by the area of the horizon, which is nothing but the Bekenstein-Hawking entropy of the black hole. This makes a connection to the generalized second law of thermodynamics (GSL).

In fact, trying to refine the connection to the GSL naturally leads to a generalization of the original Bousso's proposal [6]. Imagine a situation where we have a black hole and some matter is falling in it for a period of time. The GSL implies that the black hole entropy increment (or equivalently, its area increment) should not be less than the infalling matter entropy. Consider the past-directed null geodesics staying on the horizon as described in the above example, but now only during the interval of matter infalling. The GSL in this case is equivalent to saying that the total entropy flow on this converging lightsheet must be bounded by the area difference between two boundaries of the lightsheet. We may allow not only closed, but also open codimension two hypersurfaces.

More precisely, let hypersurface orthogonal null geodesics with affine parameter λ be generated by a null vector field k^μ, which is non expanding in the direction of k^μ i.e. $\theta \equiv \nabla_\mu k^\mu \leq 0$. The vector k^μ is either future or past directed. Denote the area of an orthogonal hypersurface at an affine parameter λ by $\mathcal{A}(\lambda)$. Then, the GCEB states that the total entropy, $S(\lambda_f, \lambda_i)$, on the lightsheet generated by null geodesics with affine parameter range $[\lambda_i, \lambda_f]$, should be bounded by $\Delta\mathcal{A}/4G = (\mathcal{A}(\lambda_i) - \mathcal{A}(\lambda_f))/4G$,

$$S(\lambda_f, \lambda_i) \leq \frac{1}{4G}(\mathcal{A}(\lambda_i) - \mathcal{A}(\lambda_f))\,. \tag{2.1}$$

The infinitesimal version [8] of this statement is

$$s(\lambda) \leq -\frac{\theta(\lambda)}{4G}\,, \tag{2.2}$$

where $s(\lambda) \equiv -s_\mu k^\mu$ (future directed), $+s_\mu k^\mu$ (past directed).

Bousso [15] showed that the GCEB actually generalizes the Bekenstein bound, $S \leq \pi M d$. There is a heuristic way to see how the Bekenstein bound emerges from the GCEB. Consider a weakly gravitating system with approximate spherical symmetry of mass M and radius R, whose center is at the position zero of a cartesian system $\{x_{1,2,3}\}$. Consider a two-surface $\{x_2^2 + x_3^2 \leq R^2\}$ of area πR^2 at $x_1 = R$, and imagine light rays orthogonal to and coming from this surface toward the system. While they are passing through the system, they are bent to converge a little bit due to gravity effect. At the position $x_1 = -R$, just behind the system, their orthogonal two surface would have a reduced radius,

$$R' \approx R - \frac{1}{2} g (\Delta t)^2 \,, \tag{2.3}$$

where $g = GM/R^2$ is the gravity acceleration at the radius R, and $\Delta t = \frac{2R}{c}$ is the traveling time of the light rays. Hence the contracted area at $x_1 = -R$ is

$$\mathcal{A}' = \pi R'^2 \approx \pi \left(R - \frac{1}{2} \frac{GM}{R^2} \left(\frac{2R}{c} \right)^2 \right)^2 \approx \pi R^2 - \frac{4\pi GMR}{c^2} \,, \tag{2.4}$$

where we assumed $R \gg 2GM/c^2$ as usual. Thus

$$\frac{\Delta \mathcal{A}}{4G} = \frac{\mathcal{A} - \mathcal{A}'}{4G} \approx \frac{\pi MR}{c^2} \,, \tag{2.5}$$

and this should bound the total entropy of the system, $S \le \pi MR$ (in $c = 1$ unit), which is the Bekenstein bound. The Bousso's analysis in ref. [15] refines the gravity effect using the Einstein's equation for an arbitrary shape of the system. The result is a generalization of the Bekenstein bound, $S \le \pi Md$, where d is now the smallest distance of any two parallel planes that can enclose the system. This generalized Bekenstein bound may become very strong for thin systems.

In refs. [6, 16, 17], the authors suggested that the infinitesimal version of the GCEB, $s(\lambda) \le -\frac{\theta(\lambda)}{4G}$, may be derived from certain assumptions,

$$i) \qquad s(0) \le -\frac{\theta(0)}{4G} \,, \tag{2.6}$$

$$ii) \qquad \frac{ds}{d\lambda} \le 2\pi T_{\mu\nu} k^\mu k^\nu \,, \tag{2.7}$$

where $T_{\mu\nu}$ is the energy-momentum tensor. Note that the gradient assumption $ii)$ should hold for both future- and past-directed null geodesics, i.e.

$$\left| \frac{ds}{d\lambda} \right| \le 2\pi T_{\mu\nu} k^\mu k^\nu \tag{2.8}$$

which implies that the weak energy condition must hold. The derivation of the GCEB from $i)$ and $ii)$ is straightforward using the Einstein's equation,

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G \, T_{\mu\nu} \,, \tag{2.9}$$

together with the Raychaudhuri's equation for null geodesics,

$$\frac{d\theta}{d\lambda} = -\frac{\theta^2}{D-2} - \sigma_{\mu\nu}\sigma^{\mu\nu} + \omega_{\mu\nu}\omega^{\mu\nu} - R_{\mu\nu} k^\mu k^\nu \,. \tag{2.10}$$

Note that for hypersurface orthogonal null geodesics, the twist $\omega_{\mu\nu} = 0$ is absent.

JHEP04(2004)061

The condition *ii*) can be taken as an infinitesimal version of the Bekenstein bound, leading to the usual Bekenstein bound when integrated [15, 16], indicating that the bound *ii*) may be, in a sense, the most fundamental principle. To show that *ii*) leads to the usual Bekenstein bound, let us consider a finite size, weakly gravitating system whose extension in x_1 direction is Δx. We call the other transverse dimensions $\vec{y}$ collectively. We next imagine almost parallel light rays (due to weak gravity) traveling along x_1 direction, and normalize their affine parameter λ such that

$$\pm k^{\mu} = \pm \frac{dx^{\mu}}{d\lambda_{\pm}} = \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x_1}\right)^{\mu}, \tag{2.11}$$

where $+k^{\mu}$ is for future-directed, while $-k^{\mu}$ is for past-directed. Thus, λ_+ (λ_-) can be identified to x_1 ($-x_1$). We let λ_+ run from x_1^i to x_1^f and λ_- from x_1^f to x_1^i, where x_1^i (x_1^f) is the starting (ending) point of the system in x_1 direction. Note that *ii*) is independent of a normalization of affine parameter. Because $s(x_1^{i,f}) = 0$, integrating *ii*) for future directed case gives

$$s(\lambda_+) \leq 2\pi \int_{x_1^i}^{x_1} d\lambda_+ \, T_{\mu\nu} k^{\mu} k^{\nu}, \tag{2.12}$$

while integrating for the past-directed equation results in

$$s(\lambda_-) \leq 2\pi \int_{\lambda_-(x_1^f)}^{\lambda_-(x_1)} d\lambda_- \, T_{\mu\nu} k^{\mu} k^{\nu} = 2\pi \int_{x_1}^{x_1^f} d\lambda_+ \, T_{\mu\nu} k^{\mu} k^{\nu}. \tag{2.13}$$

Noting that $s(\lambda_+) = s(\lambda_-) = s(x_1)$, and summing the above two equations lead to

$$s(x_1) \leq \pi \int_{x_1^i}^{x_1^f} dx_1 \, T_{\mu\nu} k^{\mu} k^{\nu}. \tag{2.14}$$

The total entropy of the system is $S = \int d\vec{y} \int dx_1 \, s(x_1, \vec{y})$, hence we get

$$S = \int d\vec{y} \int_{x_1^i}^{x_1^f} dx_1 \, s(x_1, \vec{y}) \leq \pi \Delta x \int d\vec{y} \int dx_1 \, T_{\mu\nu} k^{\mu} k^{\nu} = \pi \Delta x P_{\mu} k^{\mu}, \tag{2.15}$$

where $P_{\mu} = \int d\vec{y} \int dx_1 T_{\mu\nu} k^{\nu}$ is the total energy-momentum vector of the system. For a static system, $P_0 = M$, $P_i = 0$, we obtain the Bekenstein bound, $S \leq \pi \Delta x M$.

3. Dimensional reduction of covariant entropy bound

If the correct quantum theory of gravity is higher dimensional and it indeed has the GCEB as its fundamental aspect, we would have several extra spatial dimensions. One may expect that the GCEB in higher dimensions will dimensionally reduce to the GCEB in four dimensions for generic four dimensional observers whose energy scale is much smaller than the compactification scale. However, some trace of the higher dimensional nature of the fundamental theory may manifest itself in certain modifications of entropy bounds observed in lower dimensions. In cases where the existence of an additional dimension is only to provide some specific lower dimensional situations, like a presence of a KK U(1) gauge field, the

JHEP04(2004)061

resulting modification of entropy bounds in lower dimensions can be naturally attributed to a necessary modification of entropy bounds in lower dimensions in the presence of these specific situations. Namely dimensional reduction plays a role of a consistent tool to derive these modifications. In this section, we study the GCEB in $D+1$ spacetime dimensions, with one spatial dimension compactified on circle S^1 of coordinate size L, and describe the covariant entropy bound that is relevant to D dimensional observers. By repeating this step, we may get down to any lower dimensions from any higher dimensions.

We will consider two specific examples. In the first case, we analyze the GCEB in $D+1$ dimensions formulated in terms of two conditions i) and ii), in the space of an Einstein metric,

$$ds^2 = \tilde{g}_{MN} dx^M dx^N = K(x)^{\frac{1}{2-D}} g^E_{\alpha\beta}\, dx^\alpha dx^\beta + K(x)(dx^D)^2\,, \tag{3.1}$$

where $M, N = 0, \ldots, D$ run over $D+1$ dimensions, $\alpha, \beta = 0, \ldots, D-1$ are noncompact D dimensional indices, and x^D is the compact S^1 direction. The warp factor $K(x)$ as well as $g^E_{\mu\nu}$ is set to be independent of x^D. With the above factorization, $g^E_{\mu\nu}$ is the corresponding Einstein metric in D dimensions. Note that entropy bounds are formulated in terms of Einstein metrics. The second case we are going to study is the metric,

$$ds^2 = g^E_{\alpha\beta}\, dx^\alpha dx^\beta + (dx^D + l\, A_\alpha dx^\alpha)^2\,, \tag{3.2}$$

where again everything is set to be independent of x^D, and G is the Newton's constant in D dimensions. Here we introduce the length scale l defined by $\sqrt{16\pi G}$. In D dimensions, our system includes a KK U(1) gauge field A_α. The above normalization of A_α produces the standard gauge kinetic term. In the following, we put tilde for every relevant $D+1$ dimensional quantity.

We start from the two conditions, i) and ii), in $D+1$ dimensions,

$$i) \qquad \tilde{s}(0) \leq -\frac{\tilde{\theta}(0)}{4\tilde{G}}\,, \tag{3.3}$$

$$ii) \qquad \frac{d\tilde{s}}{d\tilde{\lambda}} \leq 2\pi \tilde{T}_{MN} q^M q^N\,, \tag{3.4}$$

which lead to the $D+1$ dimensional GCEB,

$$\tilde{s}(\tilde{\lambda}) \;\leq\; -\frac{\tilde{\theta}(\tilde{\lambda})}{4\tilde{G}}\,, \tag{3.5}$$

where q^M $(M = 0, \ldots, D)$ are hypersurface orthogonal null geodesics, $\tilde{s} = -s_M q^M$ is the entropy flow, $\tilde{\theta} = \nabla_M q^M$ is the codimension two area contraction, $\tilde{T}_{MN}$ is the energy-momentum tensor, and $\tilde{G}$ is the Newton's constant, all in $D+1$ dimensions. Note that $\tilde{\lambda}$ is the affine parameter with respect to the $D+1$ dimensional Einstein metric. Now we take a codimension two hypersurface $\tilde{B}$ which is a direct product of a codimension two hypersurface in D dimensions B and S^1 in x^D direction, i.e. $\tilde{B} = B \times S^1$. Then, we choose hypersurface orthogonal q^M of the form,

$$q^M = (q^\alpha,\, 0) \tag{3.6}$$

where q^α is independent of x^D and is hypersurface orthogonal to the D dimensional codimension two surface B. For the metrics we are going to consider, it is easy to check that $q^D = 0$ is preserved, and moreover, q^α is a null geodesic with respect to the D dimensional Einstein metric with an appropriate rescaling of its affine parameter. Thus, we may formulate a D dimensional GCEB derived from the $D+1$ dimensional GCEB for this rescaled D dimensional hypersurface orthogonal null geodesics. We will simply rewrite the conditions *i*) and *ii*) in $D+1$ dimensions in terms of correctly defined D dimensional quantities. These D dimensional conditions, *i*′) and *ii*′), have modified expressions from the original proposal *i*) and *ii*). Then, the mathematical procedure leading to $\tilde{s} \leq -\tilde{\theta}/4\tilde{G}$ from *i*) and *ii*) in $D+1$ dimensions guarantees that we must have a corresponding mathematical proof of the D dimensional version of the GCEB from the modified conditions *i*′) and *ii*′) that are obtained from dimensional reduction. Noting that the condition *ii*) (or *ii*′)) may be the most fundamental bound, the modifications in *ii*′) we have obtained in this paper should be considered seriously.

3.1 The warped metric case

We first consider the metric,

$$ds^2 = \tilde{g}_{MN}\, dx^M dx^N = K(x)^{\frac{1}{2-D}} g^E_{\alpha\beta}\, dx^\alpha dx^\beta + K(x)(dx^D)^2\,, \tag{3.7}$$

and a null geodesic vector field

$$q^M \;=\; (q^\alpha(x),\, 0)\,, \tag{3.8}$$

which is orthogonal to $\tilde{B} = B \times S^1$, and independent of x^D. From the null geodesic equation of q^M, it is easy to find that

$$k^\alpha \;\equiv\; K^{\frac{1}{2-D}}\, q^\alpha\,, \tag{3.9}$$

is a correctly normalized D dimensional null geodesic in terms of $g^E_{\mu\nu}$. In other words, the affine parameter λ with respect to $g^E_{\mu\nu}$ is given by

$$\frac{d}{d\lambda} = K^{\frac{1}{2-D}}\, \frac{d}{d\tilde{\lambda}}\,, \tag{3.10}$$

where $\tilde{\lambda}$ is the affine parameter of q^M. Thus,

$$\begin{aligned}\tilde{\theta} &\equiv \nabla_M q^M = \partial_\alpha q^\alpha + \tilde{\Gamma}^M_{M\alpha} q^\alpha \\ &= \partial_\alpha\left(K^{\frac{1}{D-2}} k^\alpha\right) + \tilde{\Gamma}^M_{M\alpha}\left(K^{\frac{1}{D-2}} k^\alpha\right),\end{aligned} \tag{3.11}$$

where $\tilde{\Gamma}$ is the affine connection of g_{MN}. Now, writing $\tilde{\Gamma}^M_{M\alpha}$ in terms of $g^E_{\mu\nu}$,

$$\begin{aligned}\tilde{\Gamma}^D_{D\alpha} &= \frac{1}{2}\partial_\alpha \log K \\ \tilde{\Gamma}^\beta_{\beta\alpha} &= \Gamma^\beta_{\beta\alpha} + \frac{D}{2(2-D)}\, \partial_\alpha \log K\,,\end{aligned} \tag{3.12}$$

where $\Gamma^\alpha_{\beta\gamma}$ is the affine connection of $g^E_{\mu\nu}$, we have

$$\tilde{\theta} = K^{\frac{1}{D-2}}\left(\partial_\alpha k^\alpha + \Gamma^\beta_{\beta\alpha} k^\alpha\right) = K^{\frac{1}{D-2}}\theta\,, \tag{3.13}$$

where $\theta = \nabla_\alpha k^\alpha$ is the area contraction of k^α in terms of $g^E_{\mu\nu}$. Also,

$$\begin{aligned}\tilde{s} &= -\tilde{s}_M\, q^M = -\tilde{s}_\alpha\, K^{\frac{1}{D-2}} k^\alpha \\ &= -\frac{s_\alpha k^\alpha}{L} K^{\frac{1}{D-2}} = \frac{s}{L} K^{\frac{1}{D-2}}\,,\end{aligned} \tag{3.14}$$

where $s_\alpha \equiv L\,\tilde{s}_\alpha$ is the D dimensional entropy current density. Using these relations, we can simply rewrite the condition $i)$ and the GCEB in $D+1$ dimensions in terms of D dimensional quantities,

$$\tilde{s}(\tilde{\lambda}) \leq -\frac{\tilde{\theta}(\tilde{\lambda})}{4\tilde{G}} \longleftrightarrow s(\lambda) \leq -\frac{\theta(\lambda)}{4G}\,, \tag{3.15}$$

where $L\,G = \tilde{G}$ is the Newton's constant in D dimensions. Henceforth, the condition $i)$ and the GCEB will trivially reduce to D dimensions.

However, a difference arises in dimensional reduction of the condition $ii)$. Recall that $\tilde{T}_{MN}$ satisfies the $D+1$ dimensional Einstein equation,

$$\tilde{R}_{MN} - \frac{1}{2}\tilde{g}_{MN}\,\tilde{R} = 8\pi\tilde{G}\,\tilde{T}_{MN}\,, \tag{3.16}$$

while the D dimensional energy-momentum tensor $T_{\alpha\beta}$ is defined to satisfy the Einstein's equation with the metric $g^E_{\mu\nu}$,

$$R_{\alpha\beta} - \frac{1}{2}g^E_{\alpha\beta}\,R = 8\pi G\,T_{\alpha\beta}\,. \tag{3.17}$$

Because q^M (k^α) is null, we have

$$\begin{aligned}8\pi\tilde{G}\,\tilde{T}_{MN}q^M q^N &= \tilde{R}_{MN}\,q^M q^N\,,\\ 8\pi G\,T_{\alpha\beta}k^\alpha k^\beta &= R_{\alpha\beta}\,k^\alpha k^\beta\,.\end{aligned} \tag{3.18}$$

Now writing $\tilde{R}_{\alpha\beta}$ in terms of the metric $g^E_{\mu\nu}$,

$$\tilde{R}_{\alpha\beta} = R_{\alpha\beta} - \frac{(D-1)}{4(D-2)}(\partial_\alpha \log K)(\partial_\beta \log K) + \frac{1}{2(D-2)}(\nabla_\gamma\nabla^\gamma \log K)g^E_{\alpha\beta}\,, \tag{3.19}$$

and $q^\alpha = K^{\frac{1}{D-2}}k^\alpha$, we get

$$8\pi\tilde{G}\,\tilde{T}_{MN}q^M q^N = K^{\frac{2}{D-2}}\left\{8\pi G\,T_{\alpha\beta}k^\alpha k^\beta - \frac{(D-1)}{4(D-2)}\left(\frac{d\log K}{d\lambda}\right)^2\right\}. \tag{3.20}$$

Also,

$$\frac{d\tilde{s}}{d\tilde{\lambda}} = \frac{K^{\frac{1}{D-2}}}{L}\frac{d}{d\lambda}\left(s\,K^{\frac{1}{D-2}}\right) = \frac{K^{\frac{2}{D-2}}}{L}\left\{\frac{ds}{d\lambda} + \frac{s}{D-2}\frac{d\log K}{d\lambda}\right\}. \tag{3.21}$$

Using these relations, the condition $ii)$, $d\tilde{s}/d\tilde{\lambda} \leq 2\pi\tilde{T}_{MN}q^M q^N$, is equivalent to

$$ii')\qquad \frac{ds}{d\lambda} \leq 2\pi T_{\alpha\beta}k^\alpha k^\beta - \frac{(D-1)}{16G(D-2)}\left(\frac{d\log K}{d\lambda}\right)^2 - \frac{s}{(D-2)}\left(\frac{d\log K}{d\lambda}\right). \tag{3.22}$$

JHEP04(2004)061

Note that once we extract $T_{\mu\nu}$ from the relevant Einstein equation, K can be an arbitrary function of x^α, which is invisible to D dimensional observers. In fact, the most interesting case is when we get the weakest condition from ii') by suitably choosing K. This is achieved by completing square of the correction terms in the rhs of ii'),

$$\Delta = -\frac{1}{16G}\frac{(D-1)}{(D-2)}\left(\frac{d\log K}{d\lambda} + \frac{8G\,s}{(D-1)}\right)^2 + \frac{4G}{(D-1)(D-2)}s^2 \,. \tag{3.23}$$

In other words,

$$ii'') \qquad \frac{ds}{d\lambda} \leq 2\pi T_{\alpha\beta}k^\alpha k^\beta + \frac{4G}{(D-1)(D-2)}\, s^2 \tag{3.24}$$

is the weakest condition we derived from dimensional reduction. We stress that the GCEB in $D+1$ dimensions guarantees that i) and ii'') should imply the GCEB in D dimensions. Imagining dimensional reduction cascade from infinite dimension to the D dimensions, we can actually strengthen ii'') to

$$ii''') \qquad \frac{ds}{d\lambda} \leq 2\pi T_{\alpha\beta}k^\alpha k^\beta + \frac{4G}{(D-2)}\, s^2 \,. \tag{3.25}$$

In fact, using the Einstein equation and the Raychaudhuri equation, we can prove this in a purely D dimensional point of view. The point is that the correction term in ii''') is taken care of by the $\theta^2/(D-2)$ term in the Raychaudhuri equation, which was previously ignored. Explicitly, using condition ii'''), the Einstein equation and $s(\lambda) \leq -\theta(\lambda)/(4G)$, one has

$$\frac{ds}{d\lambda} \leq 2\pi T_{\alpha\beta}k^\alpha k^\beta + \frac{4G}{D-2}s^2 \leq \frac{1}{4G}\left(R_{\alpha\beta}k^\alpha k^\beta + \frac{\theta^2}{D-2}\right) \leq -\frac{d\theta}{4G\,d\lambda} \tag{3.26}$$

where the Raychaudhuri equation is used for the last inequality. Combining with the initial condition, the holography $s(\lambda) \leq -\theta(\lambda)/4G$ is proved self consistently in purely D dimensional point of view. One thing to note is that we assumed $s \geq 0$ for the proof here.

3.2 The case of a KK U(1) gauge field

We next consider the metric

$$ds^2 = g^E_{\alpha\beta}dx^\alpha dx^\beta + (dx^D + l\, A_\alpha dx^\alpha)^2 \,, \tag{3.27}$$

which gives a KK U(1) gauge field A_α upon dimensional reduction. The corresponding modification of the entropy bound in D dimensions can be considered as a necessary modification of the entropy bound in the presence of a U(1) gauge interaction.

The null geodesic equations of this metric written in terms of the D dimensional Einstein metric $g^E_{\mu\nu}$ are ($q^M = (k^\alpha, q^D)$)

$$\begin{aligned} \frac{dk^\alpha}{d\lambda} + \Gamma^\alpha_{\beta\gamma}k^\beta k^\gamma &= l\, Q\, F^\alpha{}_\beta\, k^\beta \,, \\ q^D + l\, A_\alpha k^\alpha &= Q = \text{constant} \,, \end{aligned} \tag{3.28}$$

with the constraint

$$g^E_{\alpha\beta} k^\alpha k^\beta + Q^2 = 0 \,. \tag{3.29}$$

The $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$ is the usual field strength. Indices are raised or lowered by $g^E_{\mu\nu}$ always. For null geodesics with $Q = 0$, we see that k^α is a null geodesic in D dimensions with respect to $g^E_{\mu\nu}$. Moreover, we can always choose the gauge such that

$$A_\alpha k^\alpha = 0 \,. \tag{3.30}$$

In this gauge, $q^D = 0$ and we have $q^M = (k^\alpha, 0)$ as before. This enables us to take $\tilde{B} = B \times S^1$ as our codimension two hypersurface and k^α is hypersurface orthogonal to B. Hence we may formulate the D dimensional GCEB for k^α.

The $D+1$ dimensional area contraction is $\tilde{\theta} = \nabla_M q^M = \partial_\alpha k^\alpha + \tilde{\Gamma}^M_{M\alpha} k^\alpha$, and writing $D+1$ dimensional connection $\tilde{\Gamma}$ in terms of D dimensional quantities, it is straightforward to get $\tilde{\Gamma}^M_{M\alpha} = \Gamma^\beta_{\beta\alpha}$, which gives $\tilde{\theta} = \nabla_\alpha k^\alpha = \theta$. We also have

$$\tilde{s} = -\tilde{s}_M q^M = -\, \tilde{s}_\alpha k^\alpha = -\frac{s_\alpha k^\alpha}{L} = \frac{s}{L} \,, \tag{3.31}$$

where s is the entropy flux in D dimensions. Using these relations,

$$\tilde{s}(\tilde{\lambda}) \leq -\frac{\tilde{\theta}(\tilde{\lambda})}{4\tilde{G}} \;\longleftrightarrow\; s(\lambda) \;\leq\; -\frac{\theta(\lambda)}{4G} \,, \tag{3.32}$$

where $L\,G = \tilde{G}$. Thus, again the condition $i)$ and the GCEB are the same as usual.

An interesting modification arises in the condition $ii)$, however. Working out the Ricci tensor in $D+1$ dimensions in terms of D dimensional quantities,

$$\tilde{R}_{\alpha\beta} = R_{\alpha\beta} + \frac{l^2}{2}(-F_{\gamma\alpha}F^\gamma{}_\beta + A_\alpha \nabla_\gamma F_\beta{}^\gamma + A_\beta \nabla_\gamma F_\alpha{}^\gamma) + \frac{l^4}{4} A_\alpha A_\beta \, F_{\eta\gamma}F^{\eta\gamma} \,, \tag{3.33}$$

and remembering the gauge $A_\alpha k^\alpha = 0$, we have

$$\begin{aligned} 8\pi\tilde{G}\,\tilde{T}_{MN} q^M q^N &= \tilde{R}_{MN} q^M q^N = R_{\alpha\beta}k^\alpha k^\beta - 8\pi G F_{\gamma\alpha} k^\alpha F^\gamma{}_\beta k^\beta \\ &= 8\pi G (T_{\alpha\beta}k^\alpha k^\beta - F_{\gamma\alpha}k^\alpha F^\gamma{}_\beta k^\beta) \,. \end{aligned} \tag{3.34}$$

This gives us

$$2\pi \tilde{T}_{MN} q^M q^N = \frac{1}{L}\{2\pi T_{\alpha\beta}k^\alpha k^\beta - 2\pi F_{\gamma\alpha}k^\alpha \; F^\gamma{}_\beta k^\beta\} \,, \tag{3.35}$$

and the condition $ii)$ is dimensionally reduce to

$$ii') \qquad \frac{ds}{d\lambda} \leq 2\pi \, (T_{\alpha\beta} - F_{\gamma\alpha}\, F^\gamma{}_\beta) \, k^\alpha k^\beta \,. \tag{3.36}$$

We propose that this is the correct version of the Bekenstein bound in the presence of a U(1) gauge interaction. This kind of correction is not new. In the presence of electromagnetic charge, there appeared a proposal [18],

$$S \leq 2\pi R \left(E - \frac{Q^2}{2R} \right) \tag{3.37}$$

where R is the radius of the spherically symmetric system and Q is the charge. The modification appearing in our bound has the same structure because one is subtracting the electromagnetic contribution from the total energy-momentum tensor. It is clear that our proposal corresponds to the local version of (3.37).

The correction term is negative definite. To show this, let us work in a local lorentzian frame where the metric becomes flat with $g_{\mu\nu} = \mathrm{diag}(-1, 1, \ldots, 1)$. It follows that,

$$F_{\gamma\alpha}k^{\alpha}\, F^{\gamma}{}_{\beta}k^{\beta} = F_{0i}F_{0i}k_j k_j - (F_{0i}k_i)^2 + F_{ij}k_j F_{ik}k_k \geq 0\,, \tag{3.38}$$

where we used the fact $k_0^2 = k_i k_i$. Hence this part of the correction gives a tighter bound than the original proposal.

4. Physical Implication of the refined local Bekenstein bound

In the previous section, we obtained the refined version of local Bekenstein bound in D dimensions

$$\frac{ds}{d\lambda} \leq 2\pi\, T_{\mu\nu}k^{\mu}k^{\nu} + \frac{4G}{D-2}\, s^2 \tag{4.1}$$

through dimensional reduction. Once we accept the conditions i) and ii) in arbitrary dimensions for the lightlike holography, the above follows naturally as we have seen in the previous section.

First of all, the correction is of next order in the Newton's constant G. Note that the validity of the original Bekenstein bound was argued mainly for weakly gravitating systems. Since our modification is of order G, this new bound is quite consistent with previous investigations in this respect.

The second point is that now the condition allows a nontrivial saturation of the holography bound. Consider, for example, a system with spherical symmetry, which implies that the shear, $\sigma_{\mu\nu}$, is vanishing. It is clear that the lightlike holography can be saturated if the new local entropy bound is saturated and if $s(0) = -\theta(0)/(4G)$. The condition of zero expansion is no longer required.

To illustrate the above point, let us consider the AdS$_5 \times S^5$ geometry dual to the N=4 SYM theory. The AdS metric in the Poincare patch reads

$$ds^2 = R_{AdS}^2 \left(\frac{-dt^2 + dz^2 + d\vec{y}^2}{z^2} \right) \tag{4.2}$$

where $\vec{y}$ is a $D-2$ dimensional coordinate with $D = 5$ for the AdS$_5$. According to the AdS/CFT correspondence, the entropy of AdS bulk counted by the boundary area divided by $4G_5$ ought to saturate the maximal entropy of the boundary CFT [10]. Thus, for the maximal capacity of bulk entropy current, the saturation of the holography bound is necessary for the AdS/CFT correspondence to hold. The boundary of interest is located at $z = \delta$, which is related to the UV cut-off energy scale of the SYM theory by $\Lambda = 1/\delta$ via the UV/IR connection. The geodesic orthogonally generated from the boundary is described by

$$k^{\mu} = \frac{z^2}{R_{AdS}^2}\,(1,\ 1,\ \vec{0}) \tag{4.3}$$

with $k^\mu = (k^0, k^z, \vec{k}^y)$. Along the geodesic, the affine parameter λ is related to the coordinate by

$$z = -\frac{R_{AdS}^2}{\lambda} \tag{4.4}$$

with a range $(-R_{AdS}^2/\delta,\ -0)$ for $z \in (\delta, \infty)$. In the transverse space of $\vec{y}$, one has

$$\nabla_n k_m = \frac{1}{z}\,\delta_{nm} \tag{4.5}$$

where n and m are indices for the $\vec{y}$ directions. Hence $\sigma_{nm} = \omega_{nm} = 0$ and $\theta = -(D-2)\,z\,/R_{AdS}^2$. Therefore the saturation of the holography bound implies

$$s = -\frac{\theta}{4G_5} = \frac{(D-2)}{4G_5}\,\frac{z}{R_{AdS}^2} = -\frac{(D-2)}{4G_5}\,\frac{1}{\lambda}\,. \tag{4.6}$$

Since the system is time independent, the entropy current density is then

$$s^\mu = \frac{(D-2)}{4G_5}\,\frac{z}{R_{AdS}^2}\,(1,\ 0,\ \vec{0})\,, \tag{4.7}$$

which is consistent with the scale symmetry of the AdS geometry. One may check that the above entropy density saturates our local bound (4.1) as

$$\frac{ds}{d\lambda} = \frac{4G_5}{D-2}\,s^2 = \frac{(D-2)}{4G_5}\,\frac{1}{\lambda^2}\,, \tag{4.8}$$

with $T_{\mu\nu}^{AdS} k^\mu k^\nu = 0$ that follows from the fact $T_{\mu\nu}^{AdS} \sim g_{\mu\nu}$. For $\lambda \in (\lambda_i, \lambda_f)$, the integrated form of the entropy flux through the lightsheet becomes

$$S(\lambda_f, \lambda_i) \equiv \int d\vec{y} \int_{\lambda_i}^{\lambda_f} d\lambda\,\sqrt{h}\,s\,e^{\int_{\lambda_i}^{\lambda} d\lambda' \theta(\lambda')} = \frac{1}{4G_5}\int d\vec{y}\,\frac{|\lambda_i|^{D-2} - |\lambda_f|^{D-2}}{R_{AdS}^{D-2}}\,, \tag{4.9}$$

which agrees with $(\mathcal{A}_i - \mathcal{A}_f)/4G_5$. Here h is the determinant of the induced metric for the boundary surface. With the choice of $\lambda_f = -0$, the lightsheet covers the whole coordinate patch and $S = \mathcal{A}_i/4G_5$, which is the desired relation for the holography. Thus, the bulk entropy surrounded by the $z = \delta$ boundary can be consistently saturated by the Bekenstein-Hawking entropy formula. There is a nonsupersymmetric dilatonic variation of $\mathrm{AdS}_5 \times S^5$, which is called the “Janusian” background [19]. It would be interesting to ask if saturation is again possible for this background.

We now ask the implication of the refined local version to the Bekenstein bound. For simplicity, let us consider the case when s is uniform. Then by a similar computation leading to (2.15), one finds

$$S \le \pi dE + \frac{4G}{D-2}\frac{d^2}{2}A\,s^2 = \pi dE + \frac{4G}{2(D-2)}\,\frac{S^2}{A}\,. \tag{4.10}$$

The condition can be further solved in terms of S by

$$S \le \pi dE \frac{2}{1+\sqrt{1-\frac{8G}{D-2}\frac{\pi dE}{A}}} = \pi dE + \frac{4G}{2(D-2)}\,\frac{(\pi dE)^2}{A} + \cdots\,. \tag{4.11}$$

Clearly the correction is of order G/A, which comes up to our expectation. However we do not understand its physical origin. Further clarification of the refined local Bekenstein bound is necessary.

5. Discussions

In this note, we considered dimensional reduction of the lightlike holography from $D+1$ geometry of the form $M \times S^1$. With a warping factor, the local Bekenstein bound in $D+1$ dimensions leads to a more refined form of the local Bekenstein bound from the view point of the D dimensional geometry. With a KK gauge field, dimensional reduction leads to the stronger bound where the energy momentum tensor contribution is replaced by the energy momentum tensor with the electromagnetic contribution subtracted. This local version is consistent with the previously proposed modification of the Bekenstein bound in the presence of a non-vanishing electromagnetic charge.

With the order G correction of the refined local Bekenstein bound, we argued that saturation of the holography bound is now quite possible even for nonvanishing expansions. For the AdS/CFT discussed in the introduction, we showed that the regulated boundary area divided $4G$ agrees with the degrees of freedom of the boundary CFT.

There has been some challenge in finding examples violating the Bekenstein bound. Our proposal suggests that one way to find a violation of the original version is to look for possible corrections of order G. Further understanding of the refined version of the local Bekenstein bound would be very interesting.

Acknowledgments

We are grateful to Seok Kim for the contributions at the earlier stage of this work. D.B. is supported in part by Korea Research Foundation Grant KRF-2003-070-C00011. H.-U.Y. is supported in part by KOSEF R01-2003-000-10319-0.

References

[1] G. 't Hooft, *Dimensional reduction in quantum gravity*, gr-qc/9310026.

[2] L. Susskind, *The world as a hologram*, *J. Math. Phys.* **36** (1995) 6377 [hep-th/9409089].

[3] W. Fischler and L. Susskind, *Holography and cosmology*, hep-th/9806039.

[4] D. Bak and S.-J. Rey, *Cosmic holography*, *Class. and Quant. Grav.* **17** (2000) L83 [hep-th/9902173];
R. Easther and D.A. Lowe, *Holography, cosmology and the second law of thermodynamics*, *Phys. Rev. Lett.* **82** (1999) 4967 [hep-th/9902088];
G. Veneziano, *Pre-bangian origin of our entropy and time arrow*, *Phys. Lett.* **B 454** (1999) 22 [hep-th/9902126];
N. Kaloper and A.D. Linde, *Cosmology vs. holography*, *Phys. Rev.* **D 60** (1999) 103509 [hep-th/9904120].

[5] R. Bousso, *A covariant entropy conjecture*, *J. High Energy Phys.* **07** (1999) 004 [hep-th/9905177].

[6] E.E. Flanagan, D. Marolf and R.M. Wald, *Proof of classical versions of the Bousso entropy bound and of the generalized second law*, *Phys. Rev.* **D 62** (2000) 084035 [hep-th/9908070].

[7] J.D. Bekenstein, *Generalized second law of thermodynamics in black hole physics*, *Phys. Rev.* **D 9** (1974) 3292.

[8] D. Bak, *Holography with timelike bulk hypersurfaces*, *J. High Energy Phys.* **09** (2003) 069 [hep-th/0308027].

[9] J.M. Maldacena, *The large-N limit of superconformal field theories and supergravity*, *Adv. Theor. Math. Phys.* **2** (1998) 231 [hep-th/9711200].

[10] L. Susskind and E. Witten, *The holographic bound in anti-de Sitter space*, hep-th/9805114.

[11] E.K. Boyda, S. Ganguli, P. Hořava and U. Varadarajan, *Holographic protection of chronology in universes of the goedel type*, *Phys. Rev.* **D 67** (2003) 106003 [hep-th/0212087].

[12] D. Brecher, P.A. DeBoer, D.C. Page and M. Rozali, *Closed timelike curves and holography in compact plane waves*, *J. High Energy Phys.* **10** (2003) 031 [hep-th/0306190].

[13] D.K. Hong and S.D.H. Hsu, *Holography, entropy and extra dimensions*, hep-ph/0308290; *Brane world confronts holography*, hep-th/0401060.

[14] A. Karch, *Experimental tests of the holographic entropy bound*, hep-th/0311116.

[15] R. Bousso, *Light-sheets and Bekenstein's bound*, *Phys. Rev. Lett.* **90** (2003) 121302 [hep-th/0210295].

[16] R. Bousso, E.E. Flanagan and D. Marolf, *Simple sufficient conditions for the generalized covariant entropy bound*, *Phys. Rev.* **D 68** (2003) 064001 [hep-th/0305149].

[17] A. Strominger and D.M. Thompson, *A quantum Bousso bound*, hep-th/0303067.

[18] S. Hod, *Universal upper bound to the entropy of a charged system*, gr-qc/9903010; *Improved upper bound to the entropy of a charged system, II*, *Phys. Rev.* **D 61** (2000) 024023 [gr-qc/9903011];
J.D. Bekenstein and A.E. Mayo, *Black hole polarization and new entropy bounds*, *Phys. Rev.* **D 61** (2000) 024022 [gr-qc/9903002].

[19] D. Bak, M. Gutperle and S. Hirano, *A dilatonic deformation of AdS_5 and its field theory dual*, *J. High Energy Phys.* **05** (2003) 072 [hep-th/0304129];
D.Z. Freedman, C. Núñez, M. Schnabl and K. Skenderis, *Fake supergravity and domain wall stability*, hep-th/0312055.

JHEP04(2004)061